DEFORMATION THEORY OF INFINITY ALGEBRAS

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ABSTRACT. This work explores the deformation theory of algebraic structures in a very general setting. These structures include commutative, associative algebras, Lie algebras, and the infinity versions of these structures, the strongly homotopy associative and Lie algebras. In all these cases the algebra structure is determined by an element of a certain graded Lie algebra which plays the role of a differential on this algebra. We work out the deformation theory in terms of the Lie algebra of coderivations of an appropriate coalgebra structure and construct a universal infinitesimal deformation as well as a miniversal formal deformation. By working at this level of generality, the main ideas involved in deformation theory stand out more clearly.

1. INTRODUCTION

In this paper we explore the notion of deformations of algebraic structures, in a very general setting, which is applicable to any algebraic structure determined by a differential on a graded Lie algebra. Examples of such structures include associative algebras, which are determined by a differential on the Lie algebra of coderivations of the tensor coalgebra, Lie algebras, which are determined by a coderivation on the coderivations of the exterior coalgebra, as well as the infinity versions of these structures, which are determined by more general differentials than the quadratic ones which give the usual associative and Lie algebras.

Hochschild cohomology of associative algebras was first described in [15], and used to study the infinitesimal deformations of associative algebras in [9], [10]. In [9] the Gerstenhaber bracket was defined on the space of cochains of an associative algebra, which equips the space of

cochains with the structure of a graded Lie algebra. Moreover, it was shown that the cochain determining the associative algebra structure is a differential on this Lie algebra, whose homology coincides with the Hochschild cohomology of the associative algebra. Jim Stasheff discovered (see [32]) that this construction could be understood more simply by means of a natural identification of the space of cochains of an associative algebra with the coderivations of the tensor coalgebra on the underlying vector space. With this identification, the Gerstenhaber bracket coincides with the bracket of coderivations. Thus the initially mysterious existence of a Lie algebra structure on the space of cochains of an associative algebra was unraveled. In Stasheff's construction, the associative algebra structure is determined by a 2-cochain, *i.e.*, a quadratic coderivation, which has square zero, so that it is a codifferential on the *tensor* coalgebra. Infinitesimal deformations of the algebra are determined by quadratic cocycles with respect to the homology determined by the codifferential, in other words, by the second cohomology group. The connection between deformation theory and cohomology is completely transparent in this framework. It also turned out (see [29]) that replacing the quadratic codifferential by a more general codifferential determines an interesting algebraic structure, which is called an A_∞ algebra, or strongly homotopy associative algebra. A_∞ algebras first were described in [29], [30], and have appeared in both mathematics and physics. (See [11], [17], [21], [26], [16], [13].)

In a parallel manner, the notion of a Lie algebra can be described in terms of the coderivations of the *exterior* coalgebra of a vector space, with the Lie algebra structure being given by a quadratic codifferential, and the Eilenberg-Chevalley cohomology of the Lie algebra being given by the homology determined by this codifferential. In addition, the homology of a Lie algebra inherits the structure of a graded Lie algebra from the bracket of coderivations. Lie algebras generalize to L_∞ algebras (strongly homotopy Lie algebras). They first appeared in [28], and also have applications in both mathematics and physics. (See [19], [31], [18], [25], [23], [24], [1].)

The main feature of this description is that in the cases described above, the algebraic structure is determined by an element of a certain graded Lie algebra, which plays the role of the differential on this algebra, and the cohomology of the algebra is simply the homology of this differential. Infinitesimal deformations of the algebraic structure are completely determined by the cohomology, which has the structure of a graded Lie algebra. There are many other algebraic structures which fit this basic framework. For example, commutative algebras

are determined by a quadratic codifferential in the space of coderivations of the Lie coalgebra associated to the tensor coalgebra. Similarly, deformations of associative or Lie algebras preserving an invariant inner product, are determined by cyclic cohomology, which is given by a differential graded Lie algebra on a space of cyclic cochains (see [20], [25],[22]).

The notion of deformations of a Lie algebra with base given by a commutative algebra was described in [2] and used in [4] in order to study some examples of singular deformations (see also [3]). In [5] a universal infinitesimal deformation of a Lie algebra was constructed, as well as a miniversal formal deformation.Our purpose in this article is to generalize these two constructions to the general setting of an algebraic structure which is determined by a differential on a graded Lie algebra, as in the structures described above.

First, we will give a description of deformation theory in terms of the Lie algebra of coderivations of an appropriate coalgebra structure. In this general setting we will construct a universal infinitesimal deformation, as well as a miniversal formal deformation. By working at this level of generality, the main ideas involved in the construction stand out more clearly.

2. INFINITY ALGEBRAS

As a preliminary exercise, we first translate the notions of Lie algebras and associative algebras into descriptions in terms of the language of codifferentials on symmetric and tensor algebras, which allows us to give simple definitions of their generalizations into L_∞ and A_∞ algebras, as well as making it possible to describe deformation theory in a uniform manner.

Let $\mathfrak{K}$ be a field and V be a $\mathbb{Z}_2$-graded $\mathfrak{K}$-vector space. (Some may prefer a $\mathbb{Z}$-grading on V.) Then by the exterior algebra $\bigwedge V$ we mean the quotient of the (restricted) tensor algebra $\mathcal{T}(V) = \bigoplus_{n=1}^{\infty} V^n$ by the graded ideal generated by elements of the form $u \otimes v + (-1)^{uv} v \otimes u$ for homogeneous elements $u, v \in V$ (where $(-1)^{uv}$ is an abbreviation for $(-1)^{|u||v|}$, with $|u|$ denoting the parity of u). The exterior algebra may also be considered as the symmetric algebra arising from a natural $\mathbb{Z}_2 \times \mathbb{Z}$-grading on $\mathcal{T}(V)$, and it is $\mathbb{Z}_2 \times \mathbb{Z}$-graded commutative in this sense.

If σ is a permutation in Σ_n, then

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(n)} = (-1)^\sigma \epsilon(\sigma) v_1 \wedge \cdots \wedge v_n$$

where $(-1)^\sigma$ is the sign of σ, and $\epsilon(\sigma)$ is a sign depending on both σ and $v_1, \cdots, v_n$ which satisfies $\epsilon(\tau) = (-1)^{v_k v_{k+1}}$ when $\tau = (k, k+1)$ is a transposition.

In addition to the algebra structure, $\bigwedge V$ also possesses a natural $\mathbb{Z}_2 \times \mathbb{Z}$-cocommutative coalgebra structure given by

$$\Delta(v_1 \wedge \cdots \wedge v_n) =$$
$$\sum_{k=1}^{n} \sum_{\sigma \in \mathrm{Sh}(k, n-k)} (-1)^\sigma \epsilon(\sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} \otimes v_{\sigma(k+1)} \wedge \cdots \wedge v_{\sigma(n)},$$

where $\mathrm{Sh}(k, n-k)$ is the set of all unshuffles of type $(k, n-k)$, that is, permutations which are increasing on $1 \ldots k$, and on $k+1 \ldots n$.

Note that in our definition of the exterior algebra, we begin with elements of degree 1, so that $\mathfrak{K} = \bigwedge^0 V$ is not considered part of the exterior algebra. Therefore, in our construction Δ is not injective, because $\Delta(v) = 0$ for any $v \in V$; in fact, $\ker \Delta = V$.

In order to generalize Lie algebras to L_∞ algebras, it is convenient to replace the space V with its parity reversion $W = \Pi V$, and the exterior coalgebra on V with the symmetric coalgebra on W. One advantage of this construction is that the symmetric coalgebra $S(W)$ is $\mathbb{Z}_2$-graded, rather than $\mathbb{Z}_2 \times \mathbb{Z}$-graded, and the coderivations on $S(W)$ determine a $\mathbb{Z}_2$-graded Lie algebra, which is better behaved in terms of the properties of the bracket than the $\mathbb{Z}_2 \times \mathbb{Z}$-graded coderivations on $\bigwedge V$. In case the space V is not a $\mathbb{Z}_2$-graded space, that is to say that all elements in V have even parity, then W becomes a space consisting of only odd elements, and the symmetric algebra on W coincides in a straightforward manner with the exterior algebra on V, in the sense that the exterior degree of an element in $\bigwedge V$ is the same as the parity of its image in $S(W)$. Thus in the classical picture one studies simply the space $\bigwedge V$ equipped with the induced $\mathbb{Z}_2$-grading given by the $\mathbb{Z}$-grading inherited from the tensor algebra.

Let us denote the product in $S(W)$ by juxtaposition. Then the coalgebra structure on $S(W)$ is given by the rule

$$\Delta(w_1 \cdots w_n) = \sum_{k=1}^{n} \sum_{\sigma \in \mathrm{Sh}(k, n-k)} \epsilon(\sigma) w_{\sigma(1)} \cdots w_{\sigma(k)} \otimes w_{\sigma(k+1)} \cdots w_{\sigma(n)}.$$

Again the kernel of Δ is simply W, and it is injective on elements of higher degree.

There is a natural correspondence between $\mathrm{Hom}(S(W), W)$ and the $\mathbb{Z}_2$- graded Lie algebra of coderivations $\mathrm{Coder}(W)$, which is given by

extending a map $\varphi : S^k(W) \to W$ to a $\mathbb{Z}_2$-graded coderivation by the rule

$$\varphi(w_1 \cdots w_n) = \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma)\varphi(w_{\sigma(1)} \cdots w_{\sigma(k)})w_{\sigma(k+1)} \cdots w_{\sigma(n)}.$$

Moreover, if φ is any coderivation, and $\varphi_k : S^k(W) \to W$ denotes the induced maps, then φ can be recovered from these maps by the formula

$$\varphi(w_1 \cdots w_n) = \sum_{1 \leq k \leq n} \sum_{\sigma \in \mathrm{Sh}(k,n-k)} \epsilon(\sigma)\varphi_k(w_{\sigma(1)} \cdots w_{\sigma(k)})w_{\sigma(k+1)} \cdots w_{\sigma(n)}.$$

We thus can express $\varphi = \sum_{n=1}^{\infty} \varphi_n$, and say that φ_n is the degree n part of φ. Note that $\mathrm{Coder}(W)$ is actually a direct product, rather than direct sum, of its graded subspaces. In the case of Lie and associative algebras, it is conventional to consider the direct sum of the subspaces instead, but we shall not make that convention here.

If φ and ψ are two coderivations, then their Lie bracket is given by

$$[\varphi, \psi] = \varphi \circ \psi - (-1)^{\varphi\psi}\psi \circ \varphi.$$

In terms of the identification of $\mathrm{Coder}(W)$ with $\mathrm{Hom}(S(W), W)$, this bracket takes the form

$$[\varphi, \psi]_n(w_1 \cdots w_n) =$$
$$\sum_{\substack{k+l=n+1 \\ \sigma \in \mathrm{Sh}(k,n-k)}} \epsilon(\sigma)[\varphi_l(\psi_k(w_{\sigma(1)} \cdots w_{\sigma(k)})w_{\sigma(k+1)} \cdots w_{\sigma(n)})$$
$$- (-1)^{\varphi\psi}\psi_l(\varphi_k(w_{\sigma(1)} \cdots w_{\sigma(k)})w_{\sigma(k+1)} \cdots w_{\sigma(n)})].$$

A codifferential d is a coderivation whose square is zero, and in case d is odd, this is equivalent to the property $[d, d] = 0$, which can be expressed in the form

$$\sum_{\substack{k+l=n+1 \\ \sigma \in \mathrm{Sh}(k,n-k)}} \epsilon(\sigma)[d_l(d_k(w_{\sigma(1)} \cdots w_{\sigma(k)})w_{\sigma(k+1)} \cdots w_{\sigma(n)})] = 0.$$

Let $L_n = \mathrm{Hom}(S^n(W), W)$, and $L = \prod_{n=1}^{\infty} L_n = \mathrm{Coder}(W)$. An odd codifferential d_2 in L_1, called a quadratic codifferential, determines a $\mathbb{Z}_2$-graded Lie algebra structure on V. The reader can check that the codifferential property is precisely the requirement that the induced map $l_2 : V \wedge V \to V$ satisfies the $\mathbb{Z}_2$-graded Jacobi identity. In order to define a L_∞ algebra, one simply takes an arbitrary codifferential in L. This codifferential d is given by a series of maps $d_k : S^k(W) \to W$, which determine maps $l_k : \bigwedge^k V \to V$, satisfying some relations which

are the defining relations of the L_∞ algebra. Details of this construction can be obtained in [25].

It is easy to see that the map $D : L \to L$ given by $D(\varphi) = [d, \varphi]$ is a differential, equipping L with the structure of a differential graded Lie algebra. When d is quadratic, the homology given by this differential is essentially the Eilenberg-Chevalley cohomology of the Lie algebra (with coefficients in the adjoint representation). In general, the homology given by d is called the cohomology of the L_∞ algebra V. Thus both Lie and L_∞ algebras are structures on a $\mathbb{Z}_2$-graded vector space, which are determined as odd codifferentials on the symmetric coalgebra of the parity reversion of the space, and the cohomology of these algebras is simply the homology on the graded Lie algebra of coderivations of this coalgebra determined by this codifferential.

A similar picture applies for associative algebras, except that this time, we form the tensor coalgebra $\mathcal{T}(W)$ of the parity reversion W of the underlying vector space V, for which we take the $\mathbb{Z}_2$-grading corresponding to the grading of W, instead of the $\mathbb{Z}_2 \times \mathbb{Z}$-grading which would be necessary if we were to consider the coalgebra structure on $\mathcal{T}(V)$. The coproduct is given by

$$\Delta(w_1, \cdots, w_n) = \sum_{k=1}^{n-1}(w_1, \cdots, w_k) \otimes (w_{k+1}, \cdots, w_n).$$

There is a natural correspondence between $\mathrm{Hom}(\mathcal{T}(W), W)$ and the $\mathbb{Z}_2$- graded Lie algebra of coderivations $\mathrm{Coder}(W)$, which is given by extending a map $\varphi : T^k(W) \to W$ to a $\mathbb{Z}_2$-graded coderivation by the rule

$$\varphi(w_1, \cdots, w_n) = \sum_{0 \le i \le n-k} [$$
$$(-1)^{(w_1+\cdots+w_i)\varphi}(w_1, \cdots, w_i, \varphi(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n)].$$

Moreover, if φ is any coderivation, and $\varphi_k : T^k(W) \to W$ denotes the induced maps, then φ can be recovered from these maps by the formula

$$\varphi(w_1, \cdots, w_n) =$$
$$\sum_{\substack{1 \le k \le n \\ 0 \le i \le n-k}} (-1)^{(w_1+\cdots+w_i)\varphi}(w_1, \cdots, w_i, \varphi_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n).$$

If φ and ψ are two coderivations, then their Lie bracket, in terms of the identification of $\mathrm{Coder}(W)$ with $\mathrm{Hom}(\mathcal{T}(W), W)$, takes the form

$$[\varphi, \psi]_n(w_1, \cdots, w_n) = \sum_{\substack{k+l=n+1 \\ 0 \le i \le n-k}} [$$

$$(-1)^{(w_1+\cdots+w_i)\psi} \varphi_l(w_1, \cdots, w_i, \psi_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n) -$$

$$(-1)^{(\psi+w_1+\cdots+w_i)\varphi} \psi_l(w_1, \cdots, w_i, \varphi_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n)].$$

An odd codifferential d on $\mathcal{T}(W)$ satisfies the relations

$$\sum_{\substack{k+l=n+1 \\ 0 \le i \le n-k}} (-1)^{w_1+\cdots+w_i} d_l(w_1, \cdots, w_i, d_k(w_{i+1}, \cdots, w_{i+k}), w_{i+k+1}, \cdots, w_n)$$

$$= 0.$$

The relations satisfied by the induced maps $m_k : V^k \to V$ determine the structure of an A_∞ algebra on V. When d is a quadratic codifferential, one can show that the induced map m is simply an associative algebra structure on V. As in the case of an L_∞ algebra, one forms the graded Lie algebra $L = \prod_{n=1}^\infty L_n$, where $L_n = \mathrm{Hom}(T^n(V), V)$ is the subspace consisting of degree n elements in L. The differential $D(\varphi) = [\varphi, d]$ equips L with the structure of a differential graded Lie algebra, and the homology given by D is called the cohomology of the A_∞ algebra. For an associative algebra, the cohomology is simply the Hochschild cohomology of the algebra, and the bracket is the Gerstenhaber bracket.

At this point, it should be clear to the reader that from an abstract point of view there is not much difference between the construction of L_∞ and A_∞ algebras. In both cases we have a graded Lie algebra L consisting of $\mathbb{Z}_2$-graded coderivations of an appropriate coalgebra, which can be expressed as a direct product of subspaces L_n, for $n \ge 0$. It is easy to see that $[L_n, L_m] \subseteq L_{m+n-1}$. From this it follows that if d is a quadratic codifferential, then $D(L_n) \subseteq L_{n+1}$, so that the homology $H(L)$ has a decomposition $H(L) = \prod_{n=1}^\infty H_n(L)$, where $H_n(L) = \ker(d : L_n \to L_{n+1})/\mathrm{Im}(d : L_{n-1} \to L_n)$. When d is not quadratic, no such decomposition exists; instead we need to consider a natural filtration on $H(L)$. The filtration arises from the natural filtration on L given by $L^n = \prod_{i=n}^\infty L_i$, which is respected by D in the sense that $D(L^n) \subseteq L^n$. We shall discuss how the filtration on L gives rise to a natural filtration on $H(L)$ in the next section.

For Lie and associative algebras, infinitesimal deformations are determined by the second cohomology group $H^2(L)$. For infinity algebras, the entire cohomology governs the infinitesimal deformations. In fact, one interpretation of the cohomology of a Lie or associative algebra is

that it governs the infinitesimal deformations of the algebra into the appropriate infinity algebra.

Suppose now that $\mathcal{A}$ is a $\mathbb{Z}_2$-graded commutative algebra over a ground field $\mathfrak{K}$ equipped with a fixed augmentation $\epsilon : \mathcal{A} \to \mathfrak{K}$, with augmentation ideal $\mathfrak{m} = \ker(\epsilon)$. For simplicity, denote $W_{\mathcal{A}} = W \otimes \mathcal{A}$. Let $\mathcal{T}_{\mathcal{A}}(W_{\mathcal{A}})$ be the tensor algebra of $W_{\mathcal{A}}$ over $\mathcal{A}$. Then $\mathcal{T}_{\mathcal{A}}(W_{\mathcal{A}}) \cong \mathcal{T}(W) \otimes \mathcal{A}$. Similarly, if $S_{\mathcal{A}}(W_{\mathcal{A}})$ is the symmetric algebra of $W_{\mathcal{A}}$ over $\mathcal{A}$, then $S_{\mathcal{A}}(W_{\mathcal{A}}) \cong S(W) \otimes \mathcal{A}$. These natural isomorphisms respect the algebra and coalgebra structures of both sides. In what follows, let us assume for sake of definiteness that we are working with the symmetric coalgebra, and thus with the deformation theory for L_∞ algebras, but the statements and results hold true for A_∞ algebras as well, by simply replacing the symmetric coalgebra with the tensor coalgebra.

Let $L_{\mathcal{A}} = L \hat{\otimes} \mathcal{A} = \prod_{n=1}^{\infty} L_n \otimes A$ be the completed tensor product of L and $\mathcal{A}$. There is a natural identification of $L_{\mathcal{A}}$ with the space of $\mathcal{A}$-algebra coderivations of $S(W) \otimes \mathcal{A}$. Both $L_{\mathcal{A}}$ and L have the structure of $\mathbb{Z}_2$-graded $\mathcal{A}$-Lie algebras, where the augmentation determines the $\mathcal{A}$-Lie algebra structure on L. The projection $L_{\mathcal{A}} \to L$ induced by the augmentation is an $\mathcal{A}$-Lie algebra homomorphism.

A *deformation of a L_∞ structure d with base $\mathcal{A}$*, or more simply an $\mathcal{A}$-deformation of d, is defined to be an odd codifferential $\tilde{d} \in L_{\mathcal{A}}$, which maps to d under the natural projection. When $d \in L_2$, so that the structure is a Lie algebra, then an $\mathcal{A}$-deformation of the Lie algebra structure is given by an odd codifferential in $(L_{\mathcal{A}})_2$.

The algebra $\mathcal{A}$ splits canonically in the form $\mathcal{A} = \mathfrak{K} \oplus \mathfrak{m}$, which induces a splitting $L_{\mathcal{A}} = L \otimes \mathfrak{K} \oplus L \hat{\otimes} \mathfrak{m} = L \oplus L \hat{\otimes} \mathfrak{m}$; moreover, the map $L_{\mathcal{A}} \to L$ is simply the projection on the first factor. Thus if $\tilde{d}$ is an $\mathcal{A}$-deformation of L then $\tilde{d} = d + \delta$ where $\delta \in L \hat{\otimes} \mathfrak{m}$. In order for $\tilde{d}$ to be an $\mathcal{A}$-deformation, we must have $[\tilde{d}, \tilde{d}] = 0$, which is equivalent to δ satisfying the *Maurer-Cartan formula*

$$(1) \qquad\qquad D(\delta) = -\tfrac{1}{2}[\delta, \delta],$$

where $D(\delta) = [d, \delta]$.

If $\lambda : W \to W'$ is even, then it extends uniquely to a coalgebra homomorphism $S(\lambda) : S(W) \to S(W')$ by

$$S(\lambda)(w_1 \cdots w_n) = \lambda(w_1) \cdots \lambda(w_n).$$

If W and W' are equipped with Lie algebra structures d and d' respectively, then λ determines a Lie algebra homomorphism if $d' \circ S(\lambda) = S(\lambda) \circ d$. For the L_∞ case, a homomorphism is given by an arbitrary coalgebra morphism $f : S(W) \to S(W')$ satisfying $d' \circ f = f \circ d$, where

d and d' are now L_∞ algebra structures. It is not possible to translate the definition of homomorphism of L_∞ or Lie algebras into a statement about the algebras $L(W)$ and $L(W')$, but this is not surprising since homomorphisms of Lie algebras do not induce morphisms on the cohomology level.

If $f : S(W) \to S(W)$ is a coalgebra automorphism, then f induces an automorphism f^* of L, given by $f^*(\varphi) = f^{-1} \circ \varphi \circ f$, so that $d_f = f^*(d)$ is a codifferential in L if d is a codifferential. If $f = S(\tau)$ for some isomorphism $\tau : W \to W$, then $f^*(L_n) = L_n$. In particular, d_f is a quadratic codifferential when d is. For Lie algebras, we restrict our consideration of automorphisms to these degree zero maps, but for L_∞ algebras we require merely that f be even, so that it may mix the exterior degrees of elements.

Two $\mathcal{A}$-deformations $\tilde{d}$ and $\tilde{d}'$ of an L_∞ algebra are said to be *equivalent* when there is an $\mathcal{A}$-coalgebra automorphism f of $S(W) \otimes \mathcal{A}$ such that $f^*(\tilde{d}) = \tilde{d}'$, compatible with the projection $S(W) \otimes \mathcal{A} \to S(W)$. For the Lie algebra case, one requires in addition that $f = S(\gamma)$ for some isomorphism $\gamma : W_{\mathcal{A}} \to W_{\mathcal{A}}$. In terms of the decomposition $S(W) \otimes \mathcal{A} = S(W) \bigoplus S(W) \otimes \mathfrak{m}$ we can express an equivalence in the form $f = \mathrm{Id} + \lambda$, for some map $\lambda : S(W) \to S(W) \otimes \mathfrak{m}$. Since f must satisfy the condition

$$\Delta \circ f = f \otimes f \circ \Delta,$$

and Id is an automorphism, λ must satisfy the condition

$$(2) \qquad \Delta \circ \lambda = (\lambda \otimes \mathrm{Id} + \mathrm{Id} \otimes \lambda + \lambda \otimes \lambda) \circ \Delta.$$

We can also express $f^* = \mathrm{Id} + \tilde{\lambda}$, for some map $\tilde{\lambda} : L \to L \hat{\otimes} \mathfrak{m}$, and we obtain the condition

$$(3) \qquad \tilde{\lambda}[\varphi, \psi] = [\tilde{\lambda}\varphi, \psi] + [\varphi, \tilde{\lambda}\psi] + [\tilde{\lambda}\varphi, \tilde{\lambda}\psi].$$

We next define the notion of a change of base of a deformation. Suppose that $\mathcal{A}$ and $\mathcal{A}'$ are two augmented $\mathfrak{K}$-algebras with augmentation ideals $\mathfrak{m}$ and $\mathfrak{m}'$ resp. and $\tau : \mathcal{A} \to \mathcal{A}'$ is a $\mathfrak{K}$-algebra morphism. Then τ induces an $\mathcal{A}$-linear map $\tau_* = 1 \otimes \tau : S(W) \otimes \mathcal{A} \to S(W) \otimes \mathcal{A}'$ which is an $\mathcal{A}$-coalgebra morphism, that is

$$\Delta' \circ \tau_* = (\tau_* \otimes \tau_*) \circ \Delta.$$

Similarly, the induced map $\tau_* : L_{\mathcal{A}} \to L_{\mathcal{A}}{}'$ is a homomorphism of graded $\mathcal{A}$-Lie algebras. In terms of the decompositions $L_{\mathcal{A}} = L \bigoplus L \hat{\otimes} \mathfrak{m}$ and $L_{\mathcal{A}}{}' = L \bigoplus L \hat{\otimes} \mathfrak{m}'$, it is clear that $\tau_*(\varphi) = \varphi$ for any $\varphi \in L$, and $\tau_*(L \hat{\otimes} \mathfrak{m}) \subseteq L \hat{\otimes} \mathfrak{m}'$. If $\tilde{d} = d + \delta$ is an $\mathcal{A}$-deformation of d, then the *push*

out $\tau_*(\tilde{d}) = d + \tau_*(\delta)$ is an $\mathcal{A}'$-deformation of d. Furthermore, if two $\mathcal{A}$-deformations of d are equivalent, then τ_* maps them to equivalent deformations, so that τ_* induces a map between equivalence classes of $\mathcal{A}$-deformations and equivalence classes of $\mathcal{A}'$-deformations.

Two special cases of deformations with base arise, the infinitesimal deformations and the formal deformations. If $\mathcal{A}$ is a local algebra and $\mathfrak{m}^2 = 0$ then we shall call $\mathcal{A}$ an infinitesimal algebra, and an $\mathcal{A}$-deformation will be called an infinitesimal deformation. Especially interesting is the classical notion of an infinitesimal deformation, determined by the algebra $\mathcal{A} = \mathfrak{K}[t]/(t^2)$, where t is taken as an even parameter, which may be generalized to the dual algebra $\mathcal{A} = \mathfrak{K}[t, \theta]/(t^2, t\theta, \theta^2)$, where θ is taken as an odd parameter.

A formal deformation is given by taking $\mathcal{A}$ to be a complete local algebra, or a formal algebra, so that $\mathcal{A} = \overleftarrow{\lim}_{n \to \infty} A/\mathfrak{m}^n$. Then a *formal* deformation with base $\mathcal{A}$ is given by a codifferential $\tilde{d}$ on $\overleftarrow{\lim}_{n \to \infty} L \hat{\otimes} \mathcal{A}/\mathfrak{m}^n$, the classical example being $\mathcal{A} = \mathfrak{K}[[t]]$, the ring of formal power series in the even parameter t.

The main purpose of this section was to show how the notion of an $\mathcal{A}$-deformation can be given in terms of the Lie algebra $L_{\mathcal{A}}$. Deformations with base $\mathcal{A}$ are given by certain classes of codifferentials in $L_{\mathcal{A}}$, depending on the type of structure being deformed. Equivalences of deformations are given by certain classes of automorphisms of the Lie algebra structure of $L_{\mathcal{A}}$, again depending on the type of structure being deformed. We will show later how the homology of L determined by the codifferential d relates the notion of infinitesimal deformation with that of infinitesimal equivalence.

3. Filtered Topological Vector Spaces

Because we will need some topological results, we include a brief explanation of properties of *filtered topological modules*. A filtered topological space is a natural generalization of a direct product, and arises in our construction because the space L of cocycles does not have a natural decomposition as a sub-direct product of the space of cochains. The natural direct product decomposition of L does induce a filtration on the space of cocycles, which descends to a filtration on the homology. The topology of a filtered space allows the natural introduction of a dual space, the continuous dual, which is small enough to be useful in our construction.

A (decreasing) filtration on a module F is a sequence of submodules F^n satisfying $F = F^0$, $F^{n+1} \subseteq F^n$ and $\bigcap_{n=1}^{\infty} F^n = 0$. A sequence $\{x_i\}$ is said to be Cauchy if given any n, there is some m such that if

$i, j \geq m$, then $x_i - x_j \in F^n$. The space F is said to be complete if every Cauchy sequence converges, that is if there is some x such that for any n there is some m such that $x - x_i \in F^n$ if $i \geq m$. This x is unique, and is called the limit of the sequence $\{x_i\}$. This notion of convergence, whether or not F is complete, determines a topology on F, which we will call the filtered topology. The order of an element x is the largest k such that $x \in F^k$. The only element which has infinite order is zero.

A map $f : F \to G$ of two filtered topological spaces is continuous iff for given n there is some m such that $f(x) \in G^n$ whenever $x \in F^m$. To see this, note that f is continuous iff $\{f(x_i)\}$ is a Cauchy sequence whenever $\{x_i\}$ is Cauchy.

Let F_i be a subspace of F^i which projects isomorphically to F^i/F^{i+1}. Then $\bar{F} = \prod_{i=0}^{\infty} F_i$ has a natural filtration $\bar{F}^n = \prod_{i=n}^{\infty} F_i$, and is a complete filtered topological space. There is a natural map $\iota : F \to \bar{F}$ defined as follows. Let $x \in F$. Then there is a unique $x_0 \in F_0$ whose image in F_0/F_1 coincides with that of x. Then $x - x_0 \in F^1$. Continuing, one obtains a sequence of elements $x_i \in F_i$ such that $x - \sum_{i=0}^{n} x_i \in F^{n+1}$. Define $\iota(x) = \prod x_i \in \bar{F}$. The natural map ι is injective, order preserving, and continuous, and is surjective precisely when F is complete. In this case, the inverse map ι^{-1} is also continuous, so that $F \cong \bar{F}$. From this we see that a complete filtered topological space is essentially the same as a direct product.

If $F = \prod F_i$ is complete, and $\{x_i^n\}$ is a subset of F^n which projects to a basis of F^n/F^{n+1}, then any element x in F has a unique expression as an infinite sum $x = a_n^i x_i^n$ where $a_n^i \in \mathfrak{K}$ and for fixed n, only finitely many of the coefficients a_n^i are non zero. Note that x_i^n has order n. An ordered set $\{y_i\}$ which satisfies the property that every element of F can be written uniquely as an infinite sum $a^i y_i$ will be called a basis of F, it is increasing if $o(y_n) \geq o(y_{n+1})$, and strictly increasing if for any n there is some m such that $\{y_i\}_{i \geq m}$ is a basis of F^n. The basis x_i^n can be ordered in a strictly increasing manner.

If B is a subspace of F then it inherits a natural filtration $B^i = B \cap F^i$. If F is complete then the subspace B is closed in F precisely when it is complete as a filtered space. The space $H = F/B$ can be given a filtration by $H^i = \rho(F^i)$ where $\rho : F \to F/B$ is the canonical map. One obtains $H^i = F^i/B^i$ in a natural way. The requirement $\bigcap_{i=0}^{\infty} H^i = 0$ is satisfied precisely when B is closed in F. For suppose that B is not closed, and $\{b_i\}$ is a sequence in B converging to some $b \notin B$. Let $x_i = b - b_i$, so that $\rho(x_i) = \rho(b)$ for all i, but this implies that $\rho(b) \in H^i$ for all i. On the other hand, suppose that $h \neq 0 \in H^i$ for all i. Then $h = f(x_i)$ for some $x_i \in F^i$. Let $y_i = x_0 - x_i$. Then

$\rho(y_i) = 0$ for all i, so $y_i \in B$, but the sequence y_i converges to x_0, which does not lie in B. Thus the quotient space F/B is a filtered space precisely when B is closed in F. Suppose that F is complete and B is closed in F. Then we claim that $H = F/B$ is also complete. For suppose that h_i is a Cauchy sequence in H, and by taking a subsequence if necessary, we may assume that $h_i - h_{i+1} \in H^i$. Choose $x_i \in F^i$ such that $\rho(x_i) = h_i - h_{i+1}$, and $a \in F$ such that $\rho(a) = h_0$. Let $y_n = a - \sum_{i=0}^{n} x_i$. Then $y_n - y_{n+1} = x_{n+1} \in F^n$, so y_n converges to some y. Note that $\rho(y_n) = h_{n+1}$, so it follows that h_i converges to $\rho(y)$.

Next, suppose that $f : F \to G$ is a continuous map of filtered spaces, $Z = \ker f$ and $B = \operatorname{Im} f$. Then both Z and B inherit the structure of filtered spaces. It is easy to see that Z is closed in F, but it may happen that B is not closed in G. For example, if F is a filtered space which is not complete, then its image in its completion under the canonical injection is not closed. Completeness of F is also not sufficient to guarantee that the image is closed, as can be seen from the following example. Let F_0 have a countably infinite basis x_i, and $F_1 = 0$. Then F is complete for trivial reasons. Let y_i be a basis for G_i and $G = \prod G_i$ with the natural filtration. Then G is complete. Define a map $f : F \to G$ by $f(x_i) = y_i$. Then f is continuous, but its image is not closed in G. Let us say that a filtered space is of finite type if $\dim(F^n/F^{n+1}) < \infty$ for all n.

Lemma 3.1. *If F is a complete filtered space of finite type, and $f : F \to G$ is continuous, then $B = \operatorname{Im} f$ is closed in G.*

Proof. We may as well assume that $F = \prod F_i$ with the standard filtration, where each F_i is finite dimensional. If we take a basis $\{x_i^n\}$ of F_n, we can obtain a strictly increasing basis of F, and by throwing out unnecessary elements, we can choose a subsequence which spans a subspace mapping injectively to B. Let $\{y_i\}$ be the subset of B so obtained, ordered in a strictly increasing manner, because by continuity, given any n, there can only be a finite number of the x_i^m whose image has order smaller than n. Next we claim that an element y lies in B precisely if it can be written as an infinite sum of the form $y = a^k y_k$. To see this fact, first note that because the order of the x_i^n's are increasing, and y_k is the image of some $x_{i_k}^{n_k}$, we can form the element $a^k x_{i_k}^{n_k}$ which is well defined in F since it is complete, and the image of this element must be y. On the other hand, by our construction, if $y = f(a_n^i x_i^n)$, then by using the fact that for any finite combination of the $f(x_i^n)$ can be expressed in terms of the y_i, we can subtract off a linear combination $b^i y_i$ from y such that $y - b^i y_i$ is expressed as an image of terms of high order in F. This therefore must have high order

in G, and thus we can express y as the limit of a Cauchy sequence in
the y_i.   □

Now let us suppose that F has a coboundary operator d which re-
spects the filtration, that is to say that $d^2 = 0$ and $d(F^n) \subseteq F^n$ for
all n. Then if F is of finite type, the homology $H(F)$ has the natural
structure of a filtered space, and is complete if F is complete. In the
case where F is a direct product of finite dimensional vector spaces, one
immediately obtains that the homology is a complete filtered space.

Consider $\mathfrak{K}$ as a filtered space with $\mathfrak{K}^1 = 0$. Then we can form the
continuous dual space F^*. It consists of all continuous linear func-
tionals, that is all $\lambda : F \to \mathfrak{K}$ such that there is some n such that
$\lambda(x) = 0$ for all $x \in F^n$. The (continuous) dual of a filtered space
is not filtered in the sense we have described above, but does possess
an increasing filtration. To distinguish this type of space from the fil-
tered spaces given by decreasing filtrations, let us say that a space E
is *graded* if there is a sequence E_i of subspaces satisfying $E_0 = \{0\}$,
$E_n \subseteq E_{n+1}$, and $\bigcup E_n = E$. If F is filtered, then F^* is graded, with
$(F^*)_n = \{\lambda \in F^* | \lambda(F^n) = 0\}$. As a filtered space is a model of a direct
product, a graded space is a model of a direct sum. For suppose that
we choose subspaces $E^k \subseteq E_{k+1}$ such that E^k projects isomorphically
to E_{k+1}/E_k. Then there is a natural isomorphism $E \to \oplus E^k$. (This
accounts for our terminology. The usual definition of a graded space
is one given by a decomposition as a direct sum as above. Our defini-
tion generalizes the usual one.) We do not equip a graded space with
any topology, but still, its dual space has a natural filtration given by
$(E^*)^n = \{\lambda \in E^* | \lambda(E_n) = 0\}$. Moreover, the dual of a graded space is
complete.

Let us say that a $f : E \to D$ map between two graded spaces respects
the grading if given n, there is some m such that $f(E_n) \subseteq E_m$. A graded
space is said to be of *finite type* if all the E_n are finite dimensional, in
which case every map from E to a graded space respects the grading.
If $f : F \to G$ is a continuous map of filtered spaces, then it induces a
map $f^* : G^* \to F^*$, which respects the grading, giving a contravariant
functor from the category of filtered spaces with continuous maps to
the category of graded spaces with maps respecting the grading. If
$f : E \to D$ respects the grading, then $f^* : D^* \to E^*$ is continuous,
so we again a contravariant functor between graded spaces and filtered
spaces.

It is useful to note that if D is a subspace of a graded space, then D
inherits a grading given by $D_i = D \cap E_i$, E/D is graded by $(E/D)_i =$
$\text{Im}(E_i)$, and the inclusion and projection maps respect the grading.

Furthermore, any subspace of a graded space has a graded complementary subspace.

For filtered and graded spaces, a space is of finite type precisely when its dual is of finite type. Moreover, if F is a finite type filtered space, then $(F^*)^*$ is naturally identified with $\bar{F}$. For gradedspaces, the tensor product $E \otimes D$ can be graded by $(E \otimes D)_n = \sum_{p+q=n} E_p \otimes E_q$ (the sum is not direct). Similarly, the tensor product $F \otimes G$ of filtered spaces has a filtration $(F \otimes G)^n = \sum_{p+q=n} F^p \otimes G^q$. Then one obtains the useful formulae $(E \otimes D)^* = E^* \hat{\otimes} F^*$ and $(F \otimes G)^* = F^* \otimes G^*$.

When F is a finite type filtered space, and $\{x_i\}$ is a strictly increasing basis of F, then the dual basis $\{\lambda^i\}$, given by $\lambda^i(x_j) = \delta^i_j$, is a well defined basis of F^*. It is clear that λ^i is continuous, so we only need to show that any continuous linear functional λ can be represented as a sum of the λ_i. Let $a_i = \lambda(x_i)$. By continuity, the sum $a_i \lambda^i$ is well defined and coincides with λ.

Now suppose that M, N are filtered spaces, N has an increasing basis $\{y_i\}$, while M is of finite type and has an increasing basis $\{x_i\}$ with dual basis λ^i of M^*. Let η be an element of $N \otimes M^*$ which can be written in the form $\eta = a^i_j y_i \otimes \lambda^j$ for some finite sequence of elements $a^i_j \in \mathfrak{K}$. Clearly η determines a continuous map $M \to N$ by the rule $\eta(b^k x_k) = a^i_k b^k y_i$. Introduce the filtration on $N \otimes M^*$ induced by the filtration on N. Then one can form the completion $N \hat{\otimes} M^*$ of this filtration, which will have basis $\{y_i \otimes \lambda^j\}$. Any element η of the completion has a unique expression in the form $\eta = y_i \otimes \beta^i$, where $\beta^i \in M^*$ (of course, only a finite number of terms of each order can occur). When N is complete and of finite type, then η also determines a continuous map from M to N, and moreover, any continuous map is so obtained. Thus we can identify $N \otimes M^*$ with $\mathrm{Hom}(M, N)$. In general, this filtration will not be of finite type even when N has finite type and is complete. Our main interest will be in representing elements of $\mathrm{Hom}(M, N)$ by elements in $N \hat{\otimes} M^*$. When M is of finite type, and N is not necessarily complete, we still have

$$\mathrm{Hom}(M, N) \subseteq N \hat{\otimes} M^* \subseteq \bar{N} \hat{\otimes} M^* = \mathrm{Hom}(M, \bar{N}).$$

In particular, every continuous linear map has a representation as an element of $N \hat{\otimes} M^*$. Finally, note that when M is of finite type, and m^i is an increasing basis of M^*, then any element of $N \hat{\otimes} M^*$ can be expressed uniquely in the form $n_i \otimes m^i$, where n_i is an increasing sequence in N.

4. Infinitesimal Deformations

Let $\mathcal{A}$ be a $\mathbb{Z}_2$-graded commutative algebra, L a differential graded Lie algebra and $\tilde{d} = d + \delta$ be an infinitesimal $\mathcal{A}$-deformation of L. Since $\mathfrak{m}^2 = 0$, the Maurer-Cartan formula (equation 1) reduces to the cocycle condition $D(\delta) = 0$. Furthermore, an infinitesimal equivalence is of the form $f = 1 + \lambda$ where λ is a coderivation of $S(W) \otimes \mathcal{A}$, because equation 2 reduces to $\Delta \circ \lambda = (\lambda \otimes \mathrm{Id} + \mathrm{Id} \otimes \lambda) \circ \Delta$. Furthermore, if we express $f^* = \mathrm{Id} + \tilde{\lambda}$, then $\tilde{\lambda}\varphi = [\varphi, \lambda]$, since $f^{-1} = \mathrm{Id} - \lambda$, so that

$$f^*(\varphi) = (\mathrm{Id} - \lambda) \circ \varphi \circ (\mathrm{Id} + \lambda) = \varphi + \varphi \circ \lambda - \lambda \circ \varphi = \varphi + [\varphi, \lambda].$$

A trivial infinitesimal deformation is one of the form $f^*(d) = d + D(\lambda)$. Thus the (even part of the) homology $H(L) \otimes \mathfrak{m}$ classifies the equivalence classes of infinitesimal $\mathcal{A}$-deformations of an L_∞ algebra. In the case of a Lie algebra, the derivations λ giving rise to infinitesimal automorphisms are all determined by linear maps, so are elements of $L_0 \otimes \mathfrak{m}$, while the infinitesimal deformations are given by elements of $L_1 \otimes \mathfrak{m}$, so that $H_1(L) \otimes \mathfrak{m}$ classifies the equivalence classes of deformations. If $d + \delta$ and $d + \delta'$ are two $\mathcal{A}$-deformations of d, then they are equivalent precisely if $\delta - \delta'$ is a coboundary, while the condition for $d + \delta$ to be an $\mathcal{A}$-deformation is simply that $\delta \in Z(L) \hat{\otimes} \mathfrak{m}$, where $Z(L) = \ker D$ is the space of cocycles.

If $\tau : \mathcal{A} \to \mathcal{A}'$ is a morphism of infinitesimal $\mathfrak{K}$-algebras, then $\tau_* : L_{\mathcal{A}} \to L_{\mathcal{A}}'$ induces a morphism of equivalence classes of infinitesimal deformations of some fixed codifferential $d \in L$. A universal infinitesimal deformation of d is an initial object in the category of such equivalence classes. In other words, a universal infinitesimal deformation of d is given by some infinitesimal $\mathfrak{K}$-algebra $\mathcal{A}$ and $\mathcal{A}$-deformation $\tilde{d} = d + \delta$, such that if $\mathcal{A}'$ is another infinitesimal $\mathfrak{K}$-algebra, and $\tilde{d}' = d + \delta'$ is an $\mathcal{A}'$-deformation of d, then there is a unique morphism τ of infinitesimal $\mathfrak{K}$-algebras satisfying the property that $\tau_*(\tilde{d})$ is equivalent to $\tilde{d}'$.

For a $\mathbb{Z}_2$-graded space, the parity reversion of the space is given by interchanging the parity of homogeneous elements, so even elements give rise to odd elements in the parity reversion and vice versa. Let $\Pi H(L)$ denote the parity reversion of $H(L)$, and equip it with the filtration which it inherits from $H(L)$. Let $\mathfrak{m} = (\Pi H(L))^*$. Let $\mathcal{A} = \mathfrak{K} \bigoplus \mathfrak{m}$, with multiplication on $\mathfrak{m}$ defined trivially, so that $\mathcal{A}$ is an infinitesimal $\mathfrak{K}$-algebra. Let $\mu : \Pi H(L) \to Z(L)$ be a right inverse to the map $Z(L) \to \Pi H(L)$, respecting the filtrations on $\Pi H(L)$ and $Z(L)$, in other words, $\mu(\pi\delta) \in Z(L)^n$ if $\delta \in H(L)^n$. We want to represent μ as an element of $Z(L) \hat{\otimes} \mathfrak{m}$.

Let us assume that $H(L)$ is of finite type, and choose a strictly increasing basis δ_i of $H(L)$, Define $t^i \in (\Pi H(L))^*$ to be the dual basis, i.e., $t^i(\pi \delta_j) = \delta^i_j$. If we let $\mu_i = \mu(\pi \delta_i)$, then $\mu_i \hat{\otimes} t^i$ represents the map μ.

The reason we work with the parity reversion $\Pi H(L)$ instead of $H(L)$ is because we want μ to be an odd map, because we are going to define an $\mathcal{A}$-deformation $d + \mu = d + \mu_i \hat{\otimes} t^i$. So the parity reversion does the trick, because it turns an even map $H(L) \to Z(L)$ into an odd map $\Pi H(L) \to Z(L)$. Also it is immediate that $d + \mu$ is an $\mathcal{A}$-deformation, because $D(\mu) = 0$, since $\mu_i \in Z(L)$ for all i.

Moreover, if μ' is another inverse map of the canonical map $Z(L) \to H(L)$, and we express $\mu' = \mu'_i \otimes t^i$, then $\mu_i - \mu'_i = \delta(\varphi_i)$ for some φ_i. Thus we obtain that

$$\mu - \mu' = (\mu_i - \mu'_i) \otimes t^i = D(\varphi_i) \otimes t^i = D(\varphi_i \otimes t^i)$$

is a coboundary, so that the two $\mathcal{A}$-deformations are equivalent. Note that since the order of $\mu_i - \mu'_i$ is increasing, we can assume that the sequence $\{\varphi_i\}$ has only finitely many terms of any order, and therefore $\varphi_i \otimes t^i$ is a well defined element of $L \otimes \mathfrak{m}$.

Now suppose that $\mathcal{A}'$ is another infinitesimal algebra with augmentation ideal $\mathfrak{m}'$, and $d + \delta'$ is an arbitrary $\mathcal{A}'$-deformation. Then δ can be expressed in the form $\delta' = \mu_i \otimes m^i + b$, where $b \in B(L)$, and by replacing it with an equivalent deformation we can assume that $b = 0$. Define the map $f : \mathcal{A} \to \mathcal{A}'$ by $f(t^i) = m^i$. Since the λ^i are a basis for $\mathfrak{m}$, f is completely determined by this requirement. It is obvious that $f_*(d + \mu_i \otimes t^i) = d + \mu_i \otimes m^i$. It is also evident that f is unique. Thus $d + \mu$ is a universal infinitesimal deformation of d.

Finally, suppose that $\mathcal{A}$ is a $\mathfrak{K}$-algebra which may not be infinitesimal. The algebra $\mathcal{A}/\mathfrak{m}^2$ is infinitesimal, with maximal ideal $\mathfrak{m}/\mathfrak{m}^2$. Let $\tau : \mathcal{A} \to \mathcal{A}/\mathfrak{m}$ be the natural projection, and suppose that $\tilde{d} = d + \delta$ is an $\mathcal{A}$-deformation of d. Then $d + \tau_*(\delta)$ is an infinitesimal deformation, and thus determines an element $T(\delta) = [\tau_*(\delta)]$ in $H(L) \hat{\otimes} \mathfrak{m}/\mathfrak{m}^2$, which we call the differential of the deformation $\tilde{d}$.

Suppose that $\mathfrak{m}$ is graded, and define the *tangent space* of $\mathcal{A}$ by $\text{TA} = (\mathfrak{m}/\mathfrak{m}^2)^*$. Note that TA is complete. When $H(L)$ is complete and $\mathfrak{m}$ is of finite type the differential can viewed as a continuous map $T(\tilde{\delta}) : \text{TA} \to H(L)$.

In the case of an infinitesimal deformation, we can express $\delta = \delta_i \otimes m^i$, where $\delta_i \in Z(L)$, so that its differential can be expressed as $T(\tilde{d}) = [\delta] = [\delta_i] \otimes \bar{m}_i$. If the deformation is not infinitesimal then we can still

express $T(\tilde{d}) = [\delta_i \otimes \bar{m}_i]$, in terms of a decomposition $\delta = \delta_i \otimes m_i$, but the expression $[\delta_i] \otimes \bar{m}_i$ may not make sense because $\delta \notin Z(L) \otimes \mathfrak{m}$.

Our main objective is to extend the universal infinitesimal deformation to a miniversal deformation. For brevity, let us denote $\mathcal{H} = (\Pi H(L))^*$. Then the algebra $\mathcal{A} = \mathfrak{K} \bigoplus \mathcal{H}$ corresponding to the universal infinitesimal deformation can be expressed in the form $\mathcal{A} = \mathfrak{K}[[\mathcal{H}]]/(\mathcal{H})^2$. If we have any $\mathcal{A}'$-deformation, then there is a unique map from $\mathcal{A}$ to $\mathcal{A}'/(\mathfrak{m}')^2$ which takes the universal infinitesimal deformation to the induced infinitesimal deformation. We can lift the map (non-uniquely) to a map from $\mathcal{A}$ to $\mathcal{A}'$, which induces a unique algebra morphism $\mathfrak{K}[\mathcal{H}]$ to $\mathcal{A}'$. When $\mathcal{A}'$ is a formal algebra, this morphism extends uniquely to a morphism $\mathfrak{K}[[\mathcal{H}]] \to A'$. A problem that arises is that in general, the universal infinitesimal deformation does not extend to a $\mathfrak{K}[[\mathcal{H}]]$-deformation; instead, we will need to consider a quotient algebra of $\mathfrak{K}[[\mathcal{H}]]$.

The non-universality of the deformation is related to the necessity to choose a lift of $\mathcal{A}'/(\mathfrak{m}')^2$ to $\mathcal{A}'$. It is precisely this point which gives rise to the fact that there is no universal object in the category of formal deformations. Note that an induced homomorphism from a quotient of $\mathfrak{K}[[\mathcal{H}]]$ to $\mathcal{A}'$ is unique if it exists, and when $\mathcal{A}'$ is infinitesimal, this homomorphism is unique, because in this case there is no lift, and therefore no choice to make. Thus such a quotient algebra is a good candidate for a miniversal deformation. In order to construct the appropriate algebra, we will need to analyze how to extend an algebra by a trivial module. This subject was studied extensively by Harrison [14], and we shall explore this matter in the next section.

5. Harrison Cohomology

In order to understand how to construct a miniversal deformation, we shall have to consider commutative extensions of a commutative algebra by a module. We shall assume that all structures below are filtered spaces and all maps are continuous.

An extension $\mathcal{B}$ of an algebra $\mathcal{A}$ by an $\mathcal{A}$-module N is a $\mathfrak{K}$-algebra $\mathcal{B}$ together with an exact sequence of $\mathfrak{K}$-modules

$$(4) \qquad\qquad 0 \to N \xrightarrow{i} \mathcal{B} \xrightarrow{p} \mathcal{A} \to 0,$$

where p is an $\mathfrak{K}$-algebra homomorphism, and the $\mathcal{B}$-module structure on $i(N)$ is given by the $\mathcal{A}$-module structure of N by $i(n)b = i(n(p(b)))$. In particular, if we identify N with its image $i(N)$, then N is an ideal in $\mathcal{B}$ satisfying $N^2 = 0$. Let $\lambda : \mathcal{B} \to N \oplus \mathcal{A}$ be a $\mathfrak{K}$-module isomorphism such that $\lambda(n) = (n, 0)$ for $n \in N$, and such that $p = \pi_2 \circ \lambda$, where π_2

denotes the projection onto the second component. Then λ determines a product on $N \oplus \mathcal{A}$ such that $\lambda(b)\lambda(b') = \lambda(bb')$. We have

$$(n, 0)(n', 0) = \lambda(n)\lambda(n') = \lambda(nn') = (0, 0).$$

If $\lambda(b) = (0, a)$, then $p(b) = a$, so we have

$$(n, 0)(0, a) = \lambda(n)\lambda(b) = \lambda(nb) = \lambda(na) = (na, 0).$$

Finally

$$(0, a)(0, a') = (\varphi(a, a'), aa')$$

for some $\mathfrak{K}$-linear map $\varphi : a \otimes a \rightarrow N$, which has degree 0, and is graded symmetric; that is, $\varphi(a, a') = (-1)^{aa'}\varphi(a', a)$. *Two extensions $\mathcal{B}$ and $\mathcal{B}'$ of $\mathcal{A}$ by N are said to be equivalent* if there is an $\mathfrak{K}$-algebra isomorphism $f : \mathcal{B} \rightarrow \mathcal{B}'$ such that the diagram below commutes.

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
& & \| & & f \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{A} & \longrightarrow & 0
\end{array}
$$

An equivalence from $\mathcal{B}$ to $\mathcal{B}$ is said to be an *automorphism of $\mathcal{B}$ over $\mathcal{A}$*. We use a graded version of Harrison cohomology to characterize the properties of extensions.

First, consider the map φ above. It is easy to show that associativity of the multiplication is equivalent to the cocycle condition

$$a\varphi(b, c) - \varphi(ab, c) + \varphi(a, bc) - \varphi(a, b)c = 0.$$

This condition immediately yields $\varphi(1, a) = \varphi(1, 1)a$. Next, note that if $\lambda(e) = (-\varphi(1, 1), 1)$, then e is the multiplicative identity in $\mathcal{B}$ because if $\lambda(b) = (n, a)$, then

$$\lambda(eb) = (-\varphi(1, 1), 1)(n, a) =$$
$$(-\varphi(1, 1)a + n + \varphi(1, a), a) = (n, a) = \lambda(b).$$

If we define λ' by $\lambda'(b) = \lambda(b) + (\varphi(1, 1)b, 0)$, then $\lambda'(n) = (n, 0)$ and $\pi_2 \circ \lambda' = p$. Furthermore, if the product in terms of the decomposition of $\mathcal{B}$ determined by λ is given by the cocycle φ', then since $\lambda'(e) = (0, 1)$, we have

$$(0, 1) = (0, 1)(0, 1) = (\varphi'(1, 1), 1),$$

so $\varphi'(1, 1) = 0$. Thus, if $a = k + m$, $a' = k' + m'$ are elements of $\mathcal{A}$ given in terms of the decomposition $\mathcal{A} = \mathfrak{K} \oplus \mathfrak{m}$, then $\varphi'(a, a') = \varphi'(m, m')$, so φ' is completely determined by its restriction to $\mathfrak{m} \otimes \mathfrak{m}$. We shall call a cocycle φ satisfying $\varphi(1, 1)$ a reduced cocycle, and we have shown

that every extension $\mathcal{B}$ can be defined by a decomposition $B = N \oplus \mathcal{A}$, where the product is given by a reduced cocycle.

In order to relate this to Harrison cohomology, define $Ch^2(\mathcal{A}, N)$ to be the submodule of $\mathrm{Hom}(\mathfrak{m}^2, N)$ consisting of symmetric maps, and define $d_2 : Ch^2(\mathcal{A}, N) \to \mathrm{Hom}(\mathfrak{m}^3, N)$ by

$$(5) \quad d_2\varphi(m, m', m'') =$$
$$(-1)^{m\varphi}m\varphi(m', m'') - \varphi(mm', m'') + \varphi(m, m'm'') - \varphi(m, m')m''.$$

Note that even though deformations are determined only by even cocycles, we do not restrict our definitions to such elements, hence the sign appears in this definition of the coboundary operator. Also, $Ch^1(\mathcal{A}, N) = \mathrm{Hom}(\mathfrak{m}, N)$ is the space of Harrison 1-cochains, with $d_1 : Ch^1(\mathcal{A}, N) \to Ch^2(\mathcal{A}, N)$ given by

$$(6) \qquad d_1\lambda(m, m') = (-1)^{m\lambda}m\lambda(m') - \lambda(mm') + \lambda(m)m'.$$

It is easily checked that $d_1\lambda$ is graded symmetric, and that $d_1^2 = 0$. The condition $d_1\lambda = 0$ is just the derivation property, so

$$Ha^1(\mathcal{A}, N) = \ker(d_1) = \mathrm{Der}(\mathcal{A}, N).$$

Define

$$Ha^2(\mathcal{A}) = \ker(d_2)/\mathrm{Im}(d_1).$$

We show that the even part of $Ha^1(\mathcal{A}, N)$ classifies the automorphisms of an extension, while the even part of $Ha^2(\mathcal{A}, N)$ classifies the equivalence classes of extensions. Note that the even part of $Ha^k(\mathcal{A}, N)$ is not determined by just the even part of $Ha^k(\mathcal{A}, \mathfrak{K})$, which explains why we need to consider all of the cohomology, even though only even elements actually give deformations and automorphisms.

Next, note that we have already shown that an extension of $\mathcal{A}$ by N is given by a cocycle φ in $Ch^2(\mathcal{A}, N)$. We show that extensions $\mathcal{B}$ and $\mathcal{B}'$, given by cocycles φ and φ' are equivalent precisely when they differ by a coboundary. In terms of the decompositions of $\mathcal{B}$ and $\mathcal{B}'$ determined by the cocycles, an equivalence is given by a map $f : \mathcal{B} \to \mathcal{B}'$ satisfying $f(n, 0) = (n, 0)$ and $f(0, a) = (\lambda(a), a)$. If f is a homomorphism, then the two lines below are equal

$$f((0, a)(0, a')) = f(\varphi(a, a'), aa') = (\lambda(aa') + \varphi(a, a'), aa'),$$
$$f(0, a)f(0, a') = (\lambda(a), a))(\lambda(a'), a') = (\lambda(a)a' + a\lambda(a') + \varphi'(a, a'), aa'),$$

so that $\varphi = \varphi' + d\lambda$. Conversely, if $\varphi = \varphi' + d\lambda$ for some even $\lambda \in Ch^1(\mathcal{A}, N)$, then $f_\lambda(n, a) = (n + \lambda(a), a)$ defines an equivalence. Thus two cocycles are equivalent precisely when they differ by a coboundary,

and we see that the even part of $Ha^2(\mathcal{A}, N)$ classifies the extensions of $\mathcal{A}$ by N. Applying the analysis above to an automorphism of $\mathcal{B}$ over $\mathcal{A}$, we see that it is determined by an element $\lambda \in Ch^1(\mathcal{A}, N)$ satisfying the condition $d\lambda = 0$. In other words, it is a Harrison 1-cocycle. Since $Ha^1(\mathcal{A}, N)$ is precisely the space of cocycles (there are no 1-coboundaries), we see that the automorphisms of $\mathcal{B}$ over $\mathcal{A}$ are classified by $Ha^1(\mathcal{A}, N)$.

A morphism between an extension $\mathcal{B}$ of $\mathcal{A}$ by N and an extension $\mathcal{B}'$ of $\mathcal{A}$ by N' is given by a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} & \xrightarrow{\ p\ } & 0 \\
& & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\
0 & \longrightarrow & N' & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{A} & \xrightarrow{\ p'\ } & 0
\end{array} \quad ,
$$

where $g : N \to N'$ is an $\mathcal{A}$-module homomorphism and $f : \mathcal{B} \to \mathcal{B}'$ is an $\mathcal{A}$ algebra homomorphism. Given g, the homomorphism f, if it exists, is determined up to an automorphism of $\mathcal{B}'$. To see this, suppose that f and f' both satisfy the above requirements. Then $f(b) - f'(b) = n'$ for some $n' \in N'$, and if $p(b) = p(b')$, then $b - b' = n$ for some $n \in N$, so that

$$
f(b) - f'(b) - f(b') + f'(b') = f(b - b') - f'(b - b') = g(n) - g(n) = 0.
$$

Thus we can define an even $\lambda : \mathcal{A} \to N'$ such that $f(b) - f'(b) = \lambda(p(b))$. Suppose that $p(b) = a$ and $p(b') = a'$. Then we see that

$$
\begin{aligned}
\lambda(aa') = f(bb') - f'(bb') &= f(b)f(b') - f'(b)f'(b') = \\
& f(b)(f(b') - f'(b')) + (f(b) - f'(b))f'(b') = \\
& \qquad f(b)\lambda(a') + \lambda(a)f(b') = a\lambda(a') + \lambda(a)a',
\end{aligned}
$$

which shows that λ is a derivation. Now recall that the automorphism f_λ is given by $f_\lambda(b') = b' + \lambda(p'(b'))$. Finally, note that

$$
f_\lambda \circ f'(b) = f'(b) + \lambda(p'(f'(b))) = f'(b) + \lambda(p(b)) = f(b).
$$

This shows that f is uniquely defined up to an automorphism of $\mathcal{B}$, so that the equivalence classes of such mappings are determined by the map g.

Of course, we still have to determine when such a map f exists. Note that g induces a map $g_* : Ch^n(\mathcal{A}, N) \to Ch^n(\mathcal{A}, N')$, which commutes with the differential, and so induces a map $g_* : Ha^k(\mathcal{A}, N) \to Ha^k(\mathcal{A}, N')$. Then we claim that f exists precisely when $g_*(\varphi)$ is equivalent to φ', where $\mathcal{B}$ and $\mathcal{B}'$ are given by the 2-cocycles φ and φ' respectively. To see this, express $\mathcal{B} = N \oplus \mathcal{A}$ and $\mathcal{B}' = N' \oplus \mathcal{A}$, with

multiplication given by the cocycles. Suppose that $f : \mathcal{B} \to \mathcal{B}'$ exists. Then $f(n, a) = (g(n) + \lambda(a), a)$, and the homomorphism property shows that

$$(g \circ \varphi(a, a') + \lambda(aa'), aa') = f((0, a)(0, a')) = f(0, a)f(0, a') =$$
$$(\lambda(a), a)(\lambda(a'), a') = (a\lambda(a') + \lambda(a)a' + \varphi'(a, a'), aa'),$$

so that $g_*\varphi = \varphi' + d\lambda$, and they are equivalent. Conversely, if this relation holds, then $f(n, a) = (g(n) + \lambda(a), a)$ is a homomorphism satisfying the requirements. Thus it makes sense to define a morphism of extensions as a map $g : N \to N'$ which extends to a map $f : \mathcal{B} \to \mathcal{B}'$, having the required properties, since this only depends on the equivalence classes of the extensions $\mathcal{B}$ and $\mathcal{B}'$. Note that in general, there may be many such maps g.

When $\mathcal{A}$ is an augmented algebra with augmentation ideal $\mathfrak{m}$, then an *infinitesimal extension of $\mathcal{A}$ by a $\mathfrak{K}$-vector space N* is one in which the $\mathcal{A}$-module structure of N is given by the augmentation, so that in particular, $N \cdot \mathfrak{m} = 0$.

Now, let us suppose that $\mathfrak{m}$ is a graded space. Then it is natural, when N is also graded, to let $Ch^*(\mathcal{A}, N)$ be the cochains respecting the grading. When $\mathfrak{m}$ is of finite type, there is an initial object in the category of extensions of $\mathcal{A}$ by graded $\mathcal{A}$-modules N, which will play a role in the construction of the miniversal deformation of L. The inclusion $\mathrm{Hom}(\mathfrak{m}^k, N) \subseteq N \hat{\otimes}(\mathfrak{m}^k)^*$ gives rise to a map

$$Ch(\mathcal{A}, N) \to N \hat{\otimes} Ch^k(\mathcal{A}, \mathfrak{K})$$

which commutes with the differential when N is infinitesimal, and so induces a map

$$Ha^k(\mathcal{A}, N) \to N \hat{\otimes} Ha^k(\mathcal{A}, \mathfrak{K}).$$

When N is of finite type, these maps are isomorphisms. Note that $Ha^k(\mathcal{A}, \mathfrak{K})$ is a complete filtered space, which can be seen as follows. First, note that $Ch^k(\mathcal{A}, \mathfrak{K})$ is a complete subspace of $(\mathfrak{m}^k)^*$. Define a map $b : \mathfrak{m}^{k+1} \to \mathfrak{m}^k$ by

$$b(m_1, \cdots, m_{k+1}) = \sum_{i=1}^{k}(-1)^i(m_1, \cdots, m_i m_{i+1}, m_{i+2}, \cdots, m_{k+1}).$$

It is easy to see that the Harrison coboundary operator the dual of the map above, so is continuous. Thus the space of coboundaries is closed, and $Ha^k(\mathcal{A}, \mathfrak{K})$ is naturally a complete filtered space.

Let $M = Ha^2(\mathcal{A}, \mathfrak{K})^*$, equipped with the $\mathcal{A}$-module structure determined by the augmentation, and choose some order preserving $\mu :$

$Ha^2(\mathcal{A}, \mathfrak{K}) \to Ch^2(\mathcal{A}, \mathfrak{K})$ such that $\mu(\bar{\varphi}) \in \bar{\varphi}$. Then $\mu^* : \mathfrak{m} \otimes \mathfrak{m} \to M$, given by

$$(7) \qquad \mu^*(m, m')(\bar{\varphi}) = (-1)^{(m+m')\varphi}\mu(\bar{\varphi})(m, m')$$

is an even, order preserving 2 cochain. It is a 2-cocycle because

$d\mu^*(m, m', m'')(\bar{\varphi}) =$

$m\mu^*(m, m')(\bar{\varphi}) - \mu^*(mm', m'')(\bar{\varphi}) + \mu^*(m, m'm'')(\bar{\varphi}) + \mu^*(mm')m''(\bar{\varphi})$

$\quad = (-1)^{(m'+m'')\varphi}m\mu(\bar{\varphi})(m', m'') - (-1)^{(m+m'+m'')\varphi}\mu(\bar{\varphi})(mm', m'') +$

$(-1)^{(m+m'+m'')\varphi}\mu(\bar{\varphi})(m, m'm'') - (-1)^{(m+m')\varphi+m''\varphi}\mu(\bar{\varphi})(m, m')m'' =$

$$(-1)^{(m+m'+m'')\varphi}d\mu(\bar{\varphi})(m, m', m'') = 0.$$

Thus μ^* determines an extension $0 \to M \to \mathcal{C} \to \mathcal{A} \to 0$ of $\mathcal{A}$ by M. This extension does not depend, up to equivalence, on the choice of μ. To see this suppose that μ' is another choice, so that $\mu(\bar{\varphi}) - \mu'(\bar{\varphi}) = d\psi$ for some $\psi \in Ch^1(\mathcal{A})$. Note that $|\varphi| = |\psi|$. We can define $\lambda : \mathfrak{m} \to M$ by

$$\lambda(m)(\bar{\varphi}) = (-1)^{m\varphi}\psi(m).$$

Then

$d\lambda(m, m')(\bar{\varphi}) = m\lambda(m')(\bar{\varphi}) - \lambda(mm')(\bar{\varphi}) + \lambda(m)m'(\bar{\varphi}) =$

$\quad (-1)^{m'\varphi}m\psi(m') - (-1)^{(m+m')\varphi}\psi(mm') + (-1)^{(m+m')\varphi}\psi(m)m' =$

$$(-1)^{(m+m')\varphi}d\psi(m, m') = (\mu^* - \mu'^*)(m, m')(\bar{\varphi}).$$

Now we show that this extension, when $\mathfrak{m}$ is of finite type, is universal in the set of infinitesimal extensions. Let $0 \to N \to \mathcal{B} \to \mathcal{A} \to 0$ be an extension given by some (even) $\bar{\varphi} \in Ha^2(\mathcal{A}, N)$. Using the inclusion $Ha^2(\mathcal{A}, N) \subseteq N \otimes Ha^2(\mathcal{A}, k)^*$, we can express $\bar{\varphi} = n_i \otimes \bar{\varphi}^i$. Define the map $g : M \to N$ by $g(\eta) = (-1)^{\eta\varphi^i}n_i\eta(\bar{\varphi}^i)$. Let $\varphi^i = \mu(\bar{\varphi}^i)$ and define $\varphi = n_i \otimes \varphi^i$. Then φ is a cocycle representing the cohomology class $\bar{\varphi}$, and we may assume that the decomposition of $\mathcal{B} = \mathcal{A} \oplus N$ is given by the cocycle φ.

Let $\varphi \in \bar{\varphi}$ be chosen so that $\varphi^i = \mu^*(\bar{\varphi}^i)$. Then in terms of the decompositions of $\mathcal{B}$ given by φ and $\mathcal{C}$ given by μ^*, we have

$g_*(\mu^*)(m, m') = (g \circ \mu^*)(m, m') = g(\mu^*(m, m')) =$

$\quad (-1)^{n_i\mu^*(m,m')}n_i\mu^*(m, m')(\bar{\varphi}^i) =$

$$(-1)^{n_i\mu^*(m,m')+(m+m')\varphi^i}n_i\varphi^i(m, m') = \varphi(m, m'),$$

because the signs cancel owing to the fact that φ and μ^* are even maps. Any other $g : M \to N$ would determine a non equivalent cocycle, so the morphism is unique. The extension of $\mathcal{A}$ by M will be called the universal infinitesimal extension of $\mathcal{A}$. An extension of $\mathcal{A}$ by a module N is called *essential* when the cocycle $\varphi : \mathfrak{m} \otimes \mathfrak{m} \to N$ is surjective. It is useful to note that the map $\mu^* : \mathfrak{m} \otimes \mathfrak{m} \to M$ is surjective, so that it is an essential extension of $\mathcal{A}$.

Finally, we introduce the notion of a graded algebra $\mathcal{A}$, and the space of order preserving cochains. In the definition of a graded algebra, we require that $\mathcal{A}^k \cdot \mathcal{A}^l \subseteq \mathcal{A}^{k+l}$, and for augmented graded algebras, we require that $\mathfrak{m} = \mathcal{A}^1$, so that nonzero elements in the field $\mathfrak{K}$ have order zero. A typical example is a polynomial algebra, where the generators are taken to be elements of nonzero degrees. A graded module N over $\mathcal{A}$ is required to satisfy $N^k \cdot \mathcal{A}^l \subseteq N^{k+l}$. We are interested in classifying the graded extensions of $\mathcal{A}$ by a graded module N.

Since the augmentation ideal in the extended algebra is simply $\mathfrak{m}' = \mathfrak{m} \oplus N$, it is necessary that $N = N^1$. Let φ be the cocycle determining the extension. If $(0, m)$ and $(0, m')$ are elements in the extended algebra, with $m \ m' \in \mathfrak{m}$, then since $\varphi(m, m') = (0, m) \cdot (0, m')$, it follows that $\varphi(m, m') \in N^{k+l}$ if $m \in M^k$ and $m' \in M^l$. Let us say that a map $f : F \to G$ between two graded spaces is order preserving if $f(F^n) \subseteq G^n$ for all n. If we consider a 2-cochain φ to be a map $\varphi : \mathfrak{m} \otimes \mathfrak{m} \to N$, our requirement is simply that φ is order preserving. If $\varphi \in Ch^k(\mathcal{A}, N)$ is order preserving, it is easy to see that $d\varphi$ is also order preserving.

Now let us consider the universal infinitesimal extension of a graded algebra. Then since μ^* is both order preserving and surjective, it follows that the universal extension is also a graded algebra, and is thus a universal object in the category of extensions of graded algebras.

Finally, let us suppose that $0 \to M \to \mathcal{B} \to \mathcal{A} \to 0$ is the universal infinitesimal extension of $\mathcal{A}$, and that $f : \mathcal{A} \to \mathcal{A}'$ is an algebra homomorphism. Let $\mathcal{B}'$ be an infinitesimal extension of $\mathcal{A}'$ by an $\mathcal{A}'$-module N. Then there is a unique extension of the homomorphism f to a homomorphism $f' : \mathcal{B} \to \mathcal{B}'$. To see this, note that N inherits an $\mathcal{A}$-module structure through f, and moreover, if $\mathcal{B}'$ is decomposed in the form $\mathcal{B}' = \mathcal{A}' \oplus N$ using the cocycle φ', then $\varphi = \varphi' \circ (f \otimes f)$ determines a cocycle in $Ha^2(\mathcal{A}, N)$. Thus we obtain an infinitesimal extension $\mathcal{B}''$ of $\mathcal{A}$ by N, and an obvious homomorphism $\mathcal{A} \oplus N \to \mathcal{A}' \oplus N$ extending f. But there is a homomorphism from the universal extension of $\mathcal{A}$ to this extension of $\mathcal{A}$, and composition of the two homomorphisms yields the desired map. The commutative diagram below summarizes this construction:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{B}'' & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
& & \| & & f' \downarrow & & f \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{A}' & \longrightarrow & 0
\end{array}
$$

When f is surjective and N is an essential extension of $\mathcal{A}'$, then it can be seen that f' is also surjective. Note that when the algebras are graded, and f is order preserving, then we obtain an order preserving extension of this homomorphism.

An example of an infinitesimal extension which will be important to us later arises when $\mathcal{A}$ is a formal algebra, with maximal ideal $\mathfrak{m}$, and we let $\mathcal{A}_k = \mathcal{A}/\mathfrak{m}^k$. Then $N_k = \mathfrak{m}^k/\mathfrak{m}^{k+1}$ is naturally an infinitesimal $\mathcal{A}$-module, and we have an exact sequence

$$0 \to N_k \to \mathcal{A}_{k+1} \to \mathcal{A}_k \to 0,$$

expressing $\mathcal{A}_{k+1}$ as an essential, infinitesimal extension of $\mathcal{A}_k$ by N_k, when $k \geq 1$. Now consider the formal algebra $\mathcal{A}' = \mathfrak{K}[[X]]$, where $X = \mathfrak{m}/\mathfrak{m}^2$, and its corresponding quotient algebras $\mathcal{A}'_k = \mathcal{A}'/[X]^k$ ([X] is the maximal ideal in $\mathcal{A}'$). By the universal properties of the algebra $\mathcal{A}'$, one sees that the extension of $\mathcal{A}'_k$ by $N'_k = [X]^k/[X]^{k+1}$ is the universal infinitesimal extension of $\mathcal{A}'_k$. Moreover, $\mathcal{A}_2 = \mathfrak{K}[[X]]/[X]^2 = \mathcal{A}'_2$, so that the identity extends to a homomorphism $f_3 : \mathcal{A}'_3 \to A_3$, which is surjective, because both extensions are essential. Continuing on, we obtain a sequence of surjective homomorphisms $f_k : \mathcal{A}'_k \to \mathcal{A}_k$, compatible with the projections between these algebras. It follows that $\mathcal{A}$ is a quotient algebra of $\mathfrak{K}[[X]]$, and moreover, we see that if $\mathcal{A}_k = \mathfrak{K}[[X]]/I_k$ for some ideal I_k, then $[X]^k \subseteq I_k \subseteq [X]^2$.

A stronger characterization of the ideals I_k above can be obtained from some results due to Harrison, (propositions 5.1-2 in [5]).

Theorem 5.1. *Let $\mathcal{A} = \mathfrak{K}[x_1, \cdots, x_n]$ be a polynomial algebra, and $\mathfrak{m}$ be the ideal generated by $x_1, \cdots, x_n$. If I is an ideal in $\mathcal{A}$ contained in $\mathfrak{m}^2$, then*

$$Ha^2(\mathcal{A}/I, \mathfrak{K}) = (I/(\mathfrak{m} \cdot I))^*.$$

Furthermore, the universal infinitesimal extension of $\mathcal{A}/I$ is given by the exact sequence

$$0 \to I/(\mathfrak{m} \cdot I) \to \mathcal{A}/(\mathfrak{m} \cdot I) \to \mathcal{A}/I \to 0.$$

The generalization of this result to the case where $\mathcal{A} = \mathfrak{K}[[X]]$ for some graded, finite type $\mathfrak{K}$-vector space X is straightforward.

Finally, let us suppose that $0 \rightarrow N \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow 0$ is an infinitesimal extension of $\mathcal{A}$ and that N' is a closed subspace of N. Then N' is an ideal in $\mathcal{B}$, and we obtain an extension $0 \rightarrow N/N' \rightarrow \mathcal{B}/N' \rightarrow \mathcal{A} \rightarrow 0$. When the extension given by N is essential, so is the extension by N/N'.

6. Obstructions to extensions

Let $\tilde{d} = d + \delta$ be a deformation of d with base $\mathcal{A}$ and suppose that $0 \rightarrow N \rightarrow \mathcal{B} \rightarrow \mathcal{A} \rightarrow 0$ is an infinitesimal extension of $\mathcal{A}$. If we extend the base of the deformation to $\mathcal{B}$, the Maurer-Cartan formula (equation (1)) will not hold in general, but instead we obtain that

$$(8) \qquad \gamma = D(\delta) + \tfrac{1}{2}[\delta, \delta] \in L\hat{\otimes}N.$$

Moreover, γ is a cocycle in $L\hat{\otimes}N$, which can be seen as follows. First note that for an odd element in a $\mathbb{Z}_2$-graded Lie algebra, while graded antisymmetry does not force the bracket of the element with itself to vanish, nevertheless, triple brackets do vanish, *i.e.*$[\delta, [\delta, \delta]] = 0$. Next, note that $[D(\delta) + [\delta, \delta], \delta] = 0$ because the first term in the bracket is in $L\hat{\otimes}N$ and the second lies in $L\hat{\otimes}\mathfrak{m}$, and $N\mathfrak{m} = 0$. Using these facts, we obtain

$$D(D(\delta) + \tfrac{1}{2}[\delta, \delta]) = \tfrac{1}{2}D[\delta, \delta] = [D(\delta), \delta] = [D(\delta) + \tfrac{1}{2}[\delta, \delta], \delta] = 0.$$

In order to extend $\tilde{d}$ to $L_{\mathcal{B}}$, we still can add a term $\beta \in L\hat{\otimes}N$, and it is easy to see that $\tilde{d} + \beta$ is a codifferential in $L_{\mathcal{B}}$ precisely when

$$D(\beta) = -\gamma.$$

Thus the cohomology class $\bar{\gamma}$ in $H(L)\hat{\otimes}N$ determines an obstruction to extending $\tilde{d}$ to a deformation of d with base $\mathcal{B}$. Note that in the case of a Lie algebra, the obstruction actually lies in $H^3(L)\hat{\otimes}N$, but in general, we can only say that it is an even element in $H(L)\hat{\otimes}N$.

To see that the obstruction depends only on the element $f \in Ha^2(\mathcal{A})$ which determines the extension, suppose that the extension $\mathcal{B}$ is given explicitly by the Harrison cocycle $\varphi : \mathfrak{m}^2 \rightarrow N$. Writing $\delta = \delta_i \otimes m^i$, it is easily seen that $\gamma = (-1)^{\delta_j m^i}[\delta_i, \delta_j] \otimes \varphi(m^i, m^j)$. If γ' is the cocycle determined by the extension $\mathcal{B}'$ given by φ', and we express $\varphi - \varphi' = \delta(\lambda)$, for some $\lambda : \mathfrak{m} \rightarrow N$, then it follows that

$$\gamma - \gamma' = (-1)^{\delta_j m^i}[\delta_i, \delta_j] \otimes \lambda(m^i m^j) = D(\delta_i \otimes \lambda(m^i)),$$

and is thus a coboundary. Thus we have shown that the obstructions to extending the deformation to an extension of $\mathcal{A}$ by N determines a map $\mathcal{O} : Ha^2(\mathcal{A}, N) \to H(L)\hat{\otimes}N$.

Supposing that the obstruction vanishes, the element β constructed above is determined only up to a cocycle $\psi \in Z(L)\hat{\otimes}N$. Moreover, adding an odd coboundary in $B(L)\hat{\otimes}N$ produces an equivalent deformation, as in the case of infinitesimal deformations, owing to the fact that $N^2 = 0$. To see this explicitly, let $\lambda \in L\hat{\otimes}N$ be an even coderivation. Then $f = \text{Id} + \lambda$ is a coderivation of $S(W) \otimes \mathcal{B}$ (fixing $S(W) \otimes \mathcal{A}$) and $f^{-1} = \text{Id} - \lambda$, due to the fact that N is infinitesimal. Then $f_*(d + \delta + \beta) = d + \delta + \beta + D(\lambda)$. Moreover, if $d + \delta + \beta$ and $d + \delta + \beta'$ are equivalent extensions, then adding a cocycle $\psi \in Z(L)\hat{\otimes}N$ to each of them produces an equivalent deformation. Thus we obtain a transitive action of $H(L)\hat{\otimes}N$ on the equivalence classes of extensions of the deformation $\tilde{d}$ to $L\hat{\otimes}\mathcal{B}$. Note that we have not claimed that nonequivalent cocycles give rise to nonequivalent $\mathcal{B}$-deformations of d, because we only considered equivalences arising from coderivations of $S(W) \otimes \mathcal{B}$ fixing $S(W) \otimes \mathcal{A}$.

Now the set of automorphisms of $\mathcal{B}$ over $\mathcal{A}$ also acts on the equivalence classes of extensions of the deformation $\tilde{d}$ to $L\hat{\otimes}\mathcal{B}$. Let f be an automorphism of $\mathcal{B}$ over $\mathcal{A}$. Then in terms of a decomposition $\mathcal{B} = \mathcal{A} \oplus N$, with $f(n,a) = (n + \lambda(a), a)$ for some Harrison 1-cocycle λ, we have $f_*(d + \delta + \beta) = d + \delta + \beta + \lambda_*(\delta)$. Note that

$$D(\lambda_*(\delta)) = \lambda_*(D(\delta)) = \lambda_*(-D(\beta) - \frac{1}{2}[\delta, \delta]) = \lambda_*(-\frac{1}{2}[\delta, \delta]) = 0,$$

since $D(\beta) \in L\hat{\otimes}N$, $[\delta, \delta] \in L\hat{\otimes}\mathfrak{m}^2$ and λ, being a cocycle, vanishes on $\mathfrak{m}^2$. Thus $\beta + \lambda_*(\delta)$ determines another extension. If β and β' are two equivalent extensions, then $\beta + \lambda_*(\delta)$ and $\beta' + \lambda_*(\delta)$ are also equivalent, so we see that $Ha^1(\mathcal{A}, N)$ acts on the equivalence classes of extensions. We would like to show that this action is transitive, because that would show that up to an automorphism of $\mathcal{B}$, the extension of $\tilde{d}$ to $\mathcal{B}$ is unique up to equivalence. To do this, we relate the transitive action of $H(L)\hat{\otimes}N$ to the action of $Ha^1(\mathcal{A}, N)$. Since $\lambda_*(\delta) \in Z(L)\hat{\otimes}N$, it determines an element in $H(L)\hat{\otimes}N$. We investigate when this map is surjective.

The action of $\lambda \in Ha^1(\mathcal{A}, N)$ on δ can be thought of as an action of the differential $T(\delta) \in H(L)\hat{\otimes}\mathfrak{m}/\mathfrak{m}^2$ on $Ha^1(\mathcal{A}, N)$, since λ vanishes on m^2, and so acts on $\mathfrak{m}/\mathfrak{m}^2$. Let us say that the differential is surjective if we can express $T(\delta) = \delta_i \hat{\otimes} m^i$, where δ_i is an increasing spanning subset of $H(L)$, and m^i is an increasing basis of $\mathfrak{m}/\mathfrak{m}^2$. When $\mathfrak{m}/\mathfrak{m}^2$ is a dual space of a filtered space F, then this is equivalent to saying that

the associated map in $\mathrm{Hom}(F, H(L))$ is surjective. Then we claim that if N is an infinitesimal extension of $d + \delta$, and $T(\delta)$ is surjective, then the differential induces a surjective map $Ha^1(\mathcal{A}, N) \to H(L) \otimes N$. To see this, note that since $\{\delta_i\}$ spans $H(L)$, any element $H(L)\hat{\otimes}N$ can be written in the form $\delta_i\hat{\otimes}n_i$. Then define $\lambda \in Ha^1(\mathcal{A}, N)$ by $\lambda(m^i) = n^i$ (using the natural identification $Ha^1(\mathcal{A}, N) = \mathrm{Hom}(\mathfrak{m}/\mathfrak{m}^2, N)$). Then it is clear that $\lambda_*(\delta) = \delta_i\hat{\otimes}n^i$.

Our only application of the above result will be to a special sequence of extensions of the universal infinitesimal extension which satisfies $\mathfrak{m}/\mathfrak{m}^2 = (\Pi(H(L)))^* = \mathcal{H}$, as long as at each step of the way, the extensions are infinitesimal and essential. In this case, the differential is surjective, since it is given by the identity map $\Pi(H(L)) \to H(L)$.

One final result about the obstruction will prove useful in our construction of the miniversal deformation.

Theorem 6.1. *Suppose that $\tilde{d} = d + \delta$ is an $\mathcal{A}$-deformation of the codifferential d on L, and that*

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathcal{B} & \longrightarrow & \mathcal{A} & \longrightarrow & 0 \\
& & {\scriptstyle g}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\
0 & \longrightarrow & N & \longrightarrow & \mathcal{B}' & \longrightarrow & \mathcal{A} & \longrightarrow & 0
\end{array}
$$

is a diagram representing a morphism of extensions. If $\bar{\gamma} \in H(L)\hat{\otimes}M$ is the obstruction to the extension of $\tilde{d}$ to $L\hat{\otimes}\mathcal{B}$, then $\overline{g_*(\gamma)}$ is the obstruction to the extension of $\tilde{d}$ to $L\hat{\otimes}\mathcal{B}'$.

Proof. If $\delta = \delta_i\hat{\otimes}m^i$, then $\gamma = (-1)^{\delta_j m^i}[\delta_i, \delta_j]\hat{\otimes}\varphi(m^i m^j)$, where φ is the cocycle determining the extension to $\mathcal{B}$. Similarly, if $\bar{\gamma}'$ is the obstruction to extending the deformation to $\mathcal{B}'$, we can express $\gamma' = (-1)^{\delta_j m^i}[\delta_i, \delta_j]\hat{\otimes}\varphi'(m^i m^j)$, where φ' determines the extension $\mathcal{B}'$. Since $g(\varphi(m^i m^j)) = \varphi'(m^i m^j)$, the result follows immediately. $\qquad\square$

7. Construction of the Miniversal Deformation

We have assembled all of the tools for constructing the miniversal deformation in the previous two sections, so all that remains here is to tie the ideas together. Suppose that L is a differential graded Lie algebra, d is a codifferential on L, $\mathcal{A}'$ is a filtered formal algebra with maximal ideal $\mathfrak{m}'$, and that $\tilde{d} = d + \delta'$ is a formal deformation of d. Let $\mathcal{A}'_k = \mathcal{A}/(\mathfrak{m}')^k$, and $\tilde{d}'_k = d + \delta'_k$ be the induced deformation of $\tilde{d}$ on $\mathcal{A}'_k$. We will construct a formal algebra $\mathcal{A}$ with maximal ideal $\mathfrak{m}$, formal deformation $\tilde{d} = d + \delta$ with induced deformation $\tilde{d}_k$ on $L\hat{\otimes}\mathcal{A}_k$,

and homomorphisms $f_k : \mathcal{A}_k \to \mathcal{A}'_k$. Here $\mathcal{A}_k = \mathcal{A}/\mathfrak{m}^k$ are such that the diagram of extensions commutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & N_k & \longrightarrow & \mathcal{A}_{k+1} & \longrightarrow & \mathcal{A}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g_k} & & \downarrow{\scriptstyle f_{k+1}} & & \downarrow{\scriptstyle f_k} & & \\
0 & \longrightarrow & N'_k & \longrightarrow & \mathcal{A}'_{k+1} & \longrightarrow & \mathcal{A}'_k & \longrightarrow & 0
\end{array} \quad ,
$$

where $N'_k = (\mathfrak{m}')^k/(\mathfrak{m}')^{k+1}$, and in addition, $(f_k)_*(\tilde{d}_k) = \tilde{d}'_k$, so that the induced formal homomorphism $f : \mathcal{A} \to \mathcal{A}'$ satisfies $f_*(\tilde{d}) = \equiv \tilde{d}'$.

For $k = 1$, $\mathcal{A}_1 = \mathcal{A}'_1 = \mathfrak{K}$, and $\tilde{d}_1 = \tilde{d}'_1 = d$. Next, for $k = 2$, we have $N_1 = \mathcal{H}$, and the extension $\tilde{d}_1 = d + \delta_1$ is the universal infinitesimal extension of d, which comes equipped with a homomorphism $g_2 : N_1 \to N'_1$, since the extension of $\mathcal{A}'_1$ by N'_1 is infinitesimal. Now suppose that we have constructed $\mathcal{A}_k$ and the map f_k satisfying the requirements. Then consider the universal extension M of $\mathcal{A}_k$ given by Harrison cohomology. Consider the induced extension of $\mathcal{A}_k$ determined by extending it by N'_k using the homomorphism f_k to define the extension. Then we claim that the obstruction to extending $\tilde{d}_k$ to the extension of $\mathcal{A}_k$ by N'_k is the same as the obstruction to extending the deformation $\tilde{d}'_k$ to $\mathcal{A}'_{k+1}$, and therefore vanishes. But then consider the morphism of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & M & \longrightarrow & \mathcal{A}_k \oplus M & \longrightarrow & \mathcal{A}_k & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_k} & & \\
0 & \longrightarrow & N'_k & \longrightarrow & \mathcal{A}'_k \oplus N'_k & \longrightarrow & \mathcal{A}'_k & \longrightarrow & 0
\end{array} \quad ,
$$

and suppose that $\bar{\gamma} = \delta_i \hat{\otimes} n^i$ is the obstruction to extending $\tilde{d}_k$ to $\mathcal{A}_k \oplus M$. Then $g_*(\bar{\gamma}) = 0$, so that in particular $g(n^i) = 0$ for all i. Let M' be the closed subspace of M generated by the elements n_i. The map g factors through the quotient $N_k = M/M'$, and the obstruction to extending the deformation to the extension $0 \to N_k \to \mathcal{A}_k \oplus M/M' \to \mathcal{A}_k \to 0$ clearly vanishes. Let $\mathcal{A}_{k+1} = \mathcal{A}_k \oplus N_k$. Since the extension of $\mathcal{A}_k$ by N'_k is essential, we know that any two extensions of $\tilde{d}_k$ to an $\mathcal{A}_k \oplus N'_k$ are equivalent up to an automorphism. Take any extension $\tilde{d}_{k+1}$ of $\tilde{d}_k$ to $L \hat{\otimes} \mathcal{A}_{k+1}$ (again such an extension is determined up to equivalence and automorphism). After applying an automorphism η of $\mathcal{A}_k \oplus N'$, we obtain the deformation $\eta_*(f_*(\tilde{d}_{k+1}))$, which is equivalent to a deformation projecting to $\tilde{d}'_{k+1}$ by the natural map $\mathcal{A}_k \oplus N' \to \mathcal{A}'_{k+1}$. Let $f_{k+1} = \eta \circ f$, then $(f_{k+1})_*(\tilde{d}_{k+1}) \equiv \tilde{d}'_{k+1}$.

We summarize our procedure in the following theorem.

Theorem 7.1. *Suppose that L is a complete filtered $\mathbb{Z}_2$-graded Lie algebra of finite type with codifferential d. Then there is a miniversal deformation $\tilde{d} = d + \delta$ of d.*

The fact that the deformation constructed above is miniversal, rather than just versal, is immediate from the manner in which it is constructed by extensions of the universal infinitesimal deformation. Also, from some remarks made before, one can see that if $\mathcal{A}$ is the base of the miniversal deformation, then it is a quotient of $\mathfrak{K}[[\mathcal{H}]]$ by an ideal contained in $[\mathcal{H}]^2$, where $\mathcal{H} = (\Pi(H(L))^*$. Furthermore, this ideal has an increasing sequence of generators.

The proof of the existence of the miniversal deformation can be considered constructive. In fact, at each stage one obtains an obstruction γ_k, and this gives rise to some elements in $[\mathcal{H}]^k$ that need to be set equal to zero. Then one must solve the problem $D(\beta) = \gamma$, which is not such an easy problem to solve in practice. Generalized Massey products (see [8], [27]) play a role in this solution.

In a later paper, the authors will present some examples of the construction of the miniversal deformation of L_∞ algebras. It should be mentioned that even in the case of Lie algebras, the construction of the miniversal deformation may not be easy. In [5], [6] miniversal deformations of the infinite dimensional vector field Lie algebras L_1 and L_2 are constructed, and even though the cohomology is finite dimensional, the constructions are not simple. Thus, the authors feel that carrying out detailed computations here, while somewhat enlightening in the simplest cases, would overwhelm the simplicity of the general results we have presented in this paper.

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